# Some interesting things that do not exist

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# Formulae to solve polynomials

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. . . where $p = \dfrac{3c - b^2}{3}$ and $q = \dfrac{2b^3 - 9bc + 27d}{27}$.

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If $x^4 + bx^3 + cx^2 + dx + e = 0$, then

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...**and** provided $q \neq 0$! If $q = 0$, then

$$x = \pm\sqrt{-\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - r}} + \frac{b}{4}.$$

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Moreover, by the **Fundamental Theorem of Algebra**, equation (1) above **always** has between 1 and 5 distinct solutions (some possibly in $\mathbb{C}$).

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## Radical expressions

Consider the equation

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where the coefficients $a_0, a_1, \ldots a_{n-1} \in \mathbb{Q}$ are rational.

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A **radical expression** (over $\mathbb{Q}$) is a quantity that can be built up from the coefficients $a_0, a_1, \dots, a_{n-1}$, by applying the operations $+$, $-$, $\times$, $/$, and also $n$th roots $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, ....

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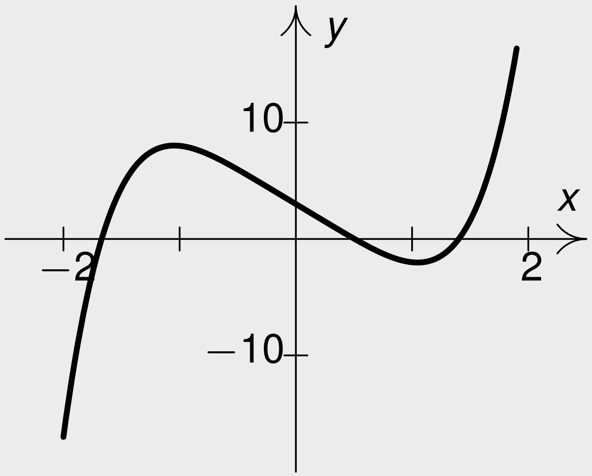
The existence of the quadratic formula implies that the solutions of every quadratic equation with rational coefficients are radical expressions. Likewise for the cubic and quartic formulae.

# There is no quintic formula

If a quintic formula existed, then the solutions of every quintic equation with rational coefficients would be radical expressions.

## Example

Consider the quintic equation $x^5 - 6x + 3 = 0$.



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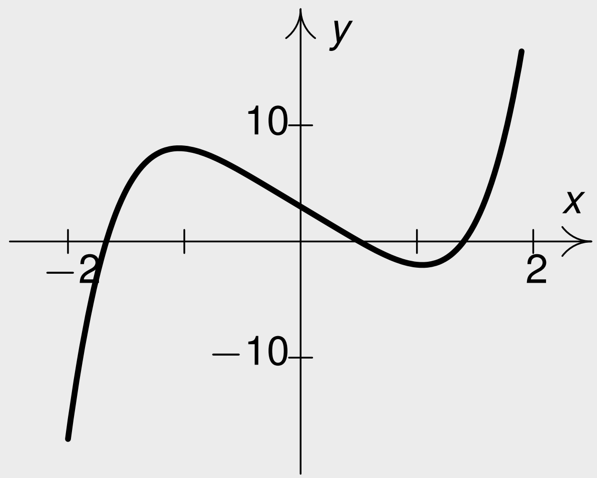


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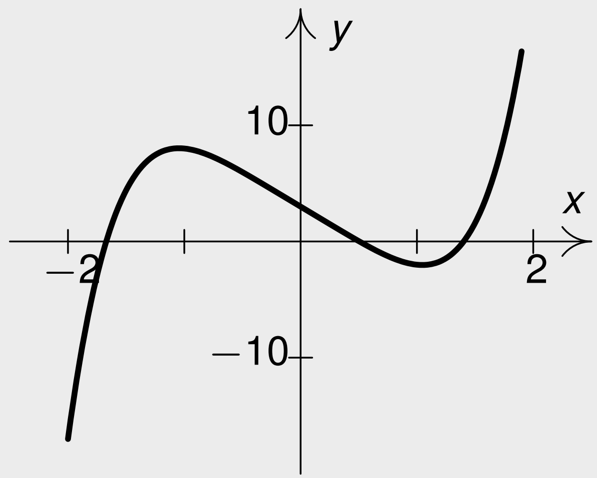


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# Polynomial equations with integer solutions

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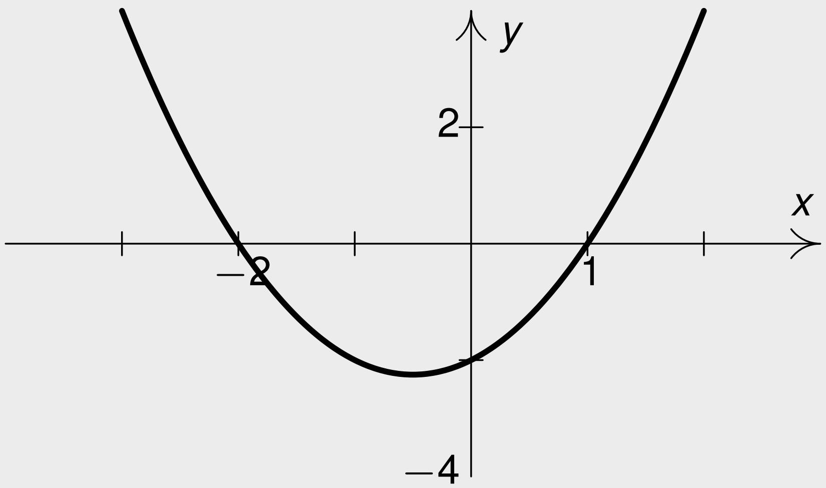
Consider the quadratic equation $x^2 + x - 2 = 0$.



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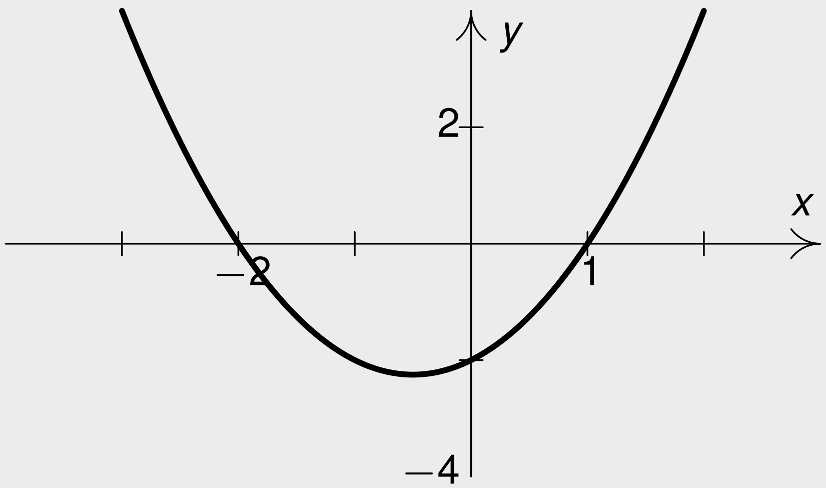


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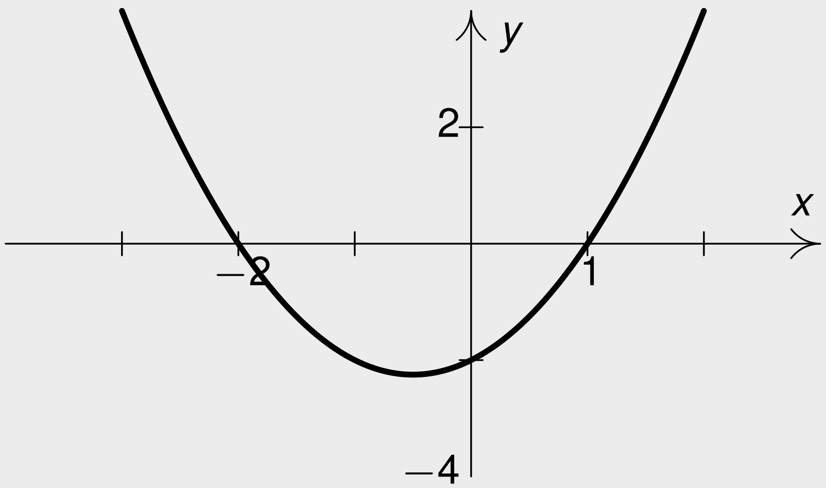


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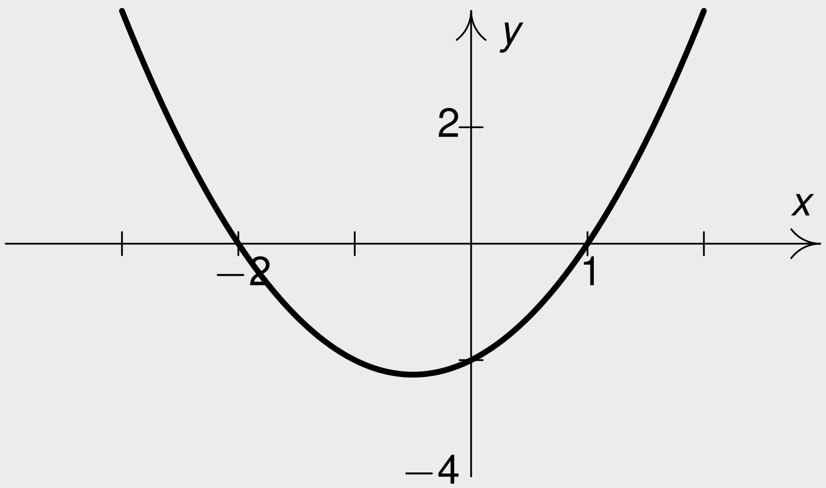


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We can consider equations in several variables too. E.g. $x^2 + y^2 - z^2 = 0$ has infinitely many integer solutions (**Pythagorean triples**).

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## Fermat's Last Theorem (Fermat 1637 (?!), Wiles 95)

Let $k \geq 3$. Then the equation

$$x^k + y^k - z^k \;=\; 0$$

has no (positive) integer solutions in $x$, $y$ and $z$.

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## Theorem (Matiyasevich 70)

No such algorithm exists!

# Turing Machines

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The memory can be regarded as an infinite tape, divided up into cells, each containing either 0 or 1.

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|---|---|---|---|---|---|---|---|---|---|---|
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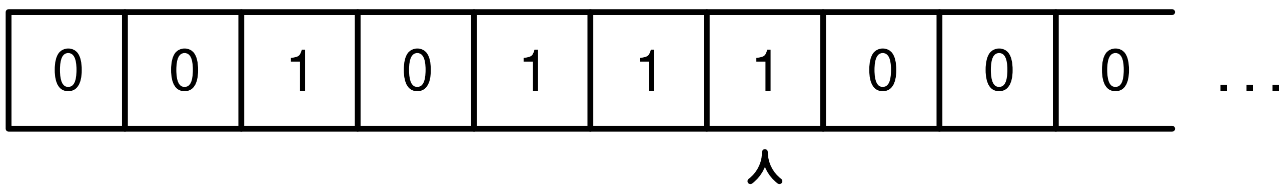


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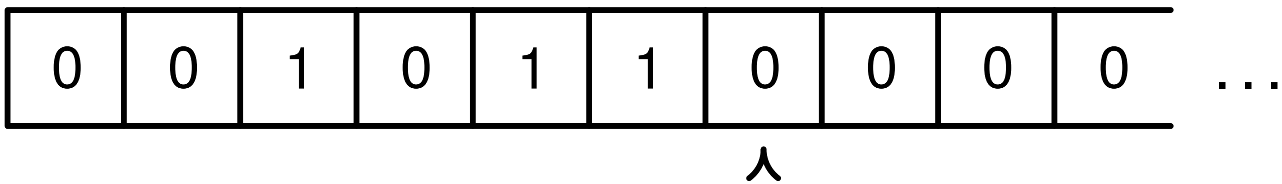


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# Turing machines as computers

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$S$ returns the output tape

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In general, $S$ computes the **successor function**

$$n \mapsto n+1.$$

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2. The function that returns the $n$th digit of $\pi$ is computable by a Turing machine.

# Turing's Halting Problem

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Define

$$h(n) = \begin{cases} 0 & \text{if } T_n \textbf{ halts}\text{, given input } n \\ 1 & \text{if } T_n \textbf{ does not halt}\text{, given input } n \end{cases}$$

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If $h$ were computable then, by the Church-Turing Thesis, it would be computable by some Turing machine $M$. There is a machine that **loops forever** given input 0, and **halts** given input 1.

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If $h$ were computable then, by the Church-Turing Thesis, it would be computable by some Turing machine $M$. There is a machine that **loops forever** given input 0, and **halts** given input 1. We combine machines to get $T$ which

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This is **even** assuming arbitrary storage capacity or execution time!

# What is true and what is false?

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Given their precision, it is reasonable to assume that any given mathematical statement can be proved to be true or false, given sufficient time and effort.

# The Continuum Hypothesis (CH)

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## Theorem (Cohen 63)

There is no proof of CH.

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So is CH true or false. . . ?!

# Dramatis personæ

al-Khwārizmī
c. 780 – c. 850

Tartaglia
1499 – 1557

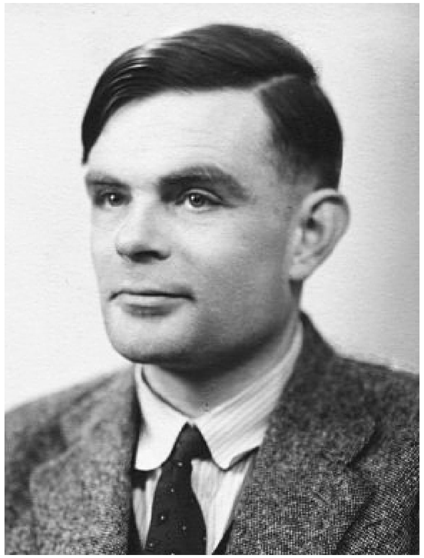
Cardano
1501 – 1576

# Dramatis personæ

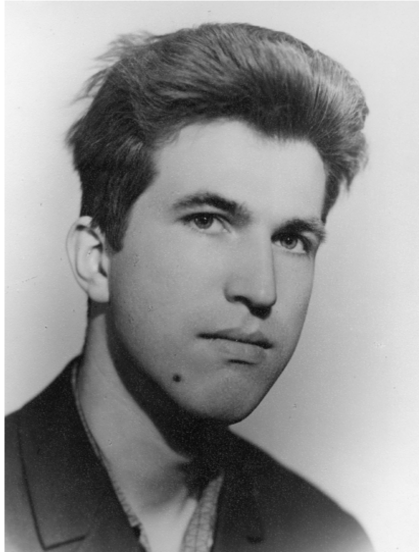
Abel
1802 – 1829

Galois
1811 – 1832

# Dramatis personæ



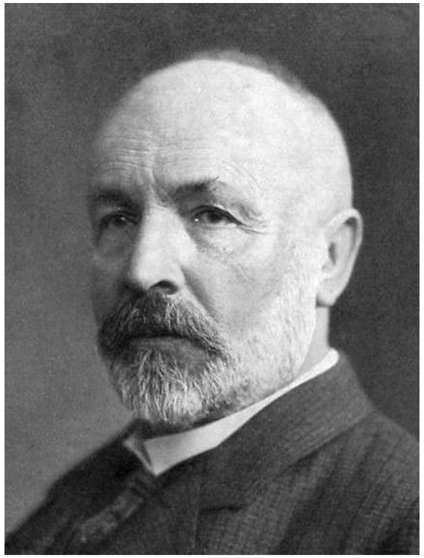
Turing
1912 – 1954



Matiyasevich
1947 –

Wiles
1953 –

# Dramatis personæ

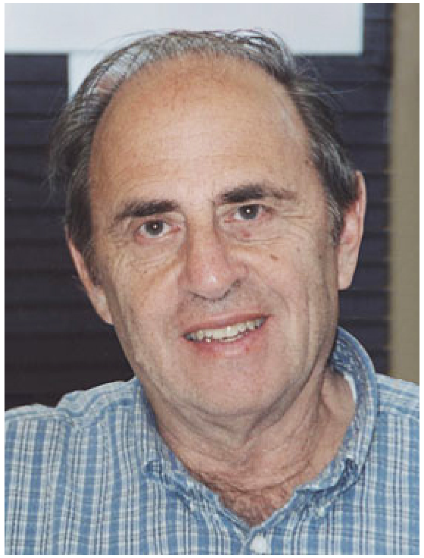


Cantor
1845 – 1918

Hilbert
1862 – 1943

# Dramatis personæ

Gödel
1906 – 1978



Cohen
1934 – 2007